## § 1.7 Linear Independence

Given a set of vectors $\{v_1, v_2, \cdots, v_\ell\}$ of $\mathbb{R}^n$,

- We say $\{v_1, \cdots, v_\ell\}$ is linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_\ell v_\ell = 0$$

has only the trivial solution $(x = 0)$.

- We say $\{v_1, \cdots, v_\ell\}$ is <u>linearly dependent</u> if there is a nontrivial solution to the equation above. I.e. there are real numbers $c_1, \cdots, c_\ell$ not all zero such that

$$c_1 v_1 + \cdots + c_\ell v_\ell = 0$$

such an equation is called an <u>equation (or relation) of linear dependence</u>.

<u>Remark</u>

- Let $v$ be a vector. The set $\{v\}$ is linearly independent if and only if $v \neq 0$.
- A set of two vectors $\{v_1, v_2\}$ is linearly independent if and only if neither of the two vectors is a multiple of the other.

Using the terminology from last time:

## Theorem

$\{v_1, \dots, v_\ell\}$ is linearly independent if and only if the matrix equation $Ax = 0$ has only the trivial solution where $A = [v_1 | v_2 | \cdots | v_\ell]$.

## Example

Determine if $v_1, v_2, v_3$ are linearly independent where

$$v_1 = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -4 \\ 4 \\ 8 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

Solve $Ax = 0$

$$\left[\begin{array}{ccc|c} 3 & -4 & 6 & 0 \\ 4 & 4 & 8 & 0 \\ 6 & 8 & 12 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 \to} \left[\begin{array}{ccc|c} 1 & -4/3 & 2 & 0 \\ 4 & 4 & 8 & 0 \\ 6 & 8 & 12 & 0 \end{array}\right] \xrightarrow[-6R_1 + R_3 \to R_3]{-4R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -4/3 & 2 & 0 \\ 0 & 28/3 & 0 & 0 \\ 0 & 16 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{3}{28}R_2} \left[\begin{array}{ccc|c} 1 & -4/3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 16 & 0 & 0 \end{array}\right] \xrightarrow[-16R_2 + R_3 \to R_3]{\frac{4}{3}R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3$ free so infinitely many nontrivial solutions!

$$\begin{cases} x_1 = -2x_3 \\ x_2 = 0 \\ x_3 = \text{free} \end{cases}$$

solution set $= \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

No! $\{v_1, v_2, v_3\}$ is not linearly independent.

$$-2v_1 + 0v_2 + v_3 = 0$$

## Theorem

The set $\{v_1, \dots, v_\ell\}$ where $\ell \geq 2$ is linearly dependent if and only if at least one of the vectors $v_1, \dots, v_\ell$ is a linear combination of the others

## Proof

By definition of linear dependence

$$c_1 v_1 + c_2 v_2 + \cdots + c_\ell v_\ell = 0$$

where at least one of the $c_1, \dots, c_\ell$ is nonzero.

Say $c_i \neq 0$ for some $1 \leq i \leq \ell$. Then

$$c_1 v_1 + \cdots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \cdots + c_\ell v_\ell = -c_i v_i$$

so

$$-\frac{c_1}{c_i} v_1 + \cdots + \left(\frac{c_{i-1}}{-c_i}\right) v_{i-1} + \left(\frac{c_{i+1}}{-c_i}\right) v_{i+1} + \cdots + \left(\frac{c_\ell}{-c_i}\right) v_\ell = v_i$$

so $v_i$ is a linear combination of $\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_\ell\}$

## Warning!

This theorem does not say ~~at~~ every vector is a linear combination of ~~them~~ others, but that at least one is.

For example $\{v_1, v_2, v_3\}$ is linearly dependent where

$$v_1 = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -4 \\ 4 \\ 8 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

but $v_2$ is not a linear combination of $v_1$ and $v_3$.

check this!

As a follow-up to the previous theorem, we have the following result. (Useful for written Hw §1.7 Pg 65 #32)

Theorem

Suppose $\{v_1, \dots, v_\ell\}$ is a linearly independent set. Then (nonzero) $w$ is in $\text{span}\{v_1, \dots, v_\ell\}$ if and only if $\{v_1, \dots, v_\ell, w\}$ is linearly dependent.

Proof

$\Rightarrow$] Suppose $w$ in $\text{span}\{v_1, \dots, v_\ell\}$. Then

$$c_1 v_1 + \dots + c_\ell v_\ell = w \quad \text{for some (not all ~~nonzero~~ zero) } c_1, \dots, c_\ell \text{ in } \mathbb{R}$$

Then $c_1 v_1 + \dots + c_\ell v_\ell - 1 \cdot w = 0$

so $\{v_1, \dots, v_\ell, w\}$ is linearly dependent since not all of $c_1, \dots, c_\ell$ are zero.

$\Leftarrow$] Suppose $\{v_1, \dots, v_\ell, w\}$ is linearly dependent. Then

$$c_1 v_1 + \dots + c_\ell v_\ell + d w = 0 \quad \text{for } c_1, \dots, c_\ell, d \text{ in } \mathbb{R} \text{ not all zero}$$

Now if $d = 0$, then $c_1 = \dots = c_\ell = 0$ too since $\{v_1, \dots, v_\ell\}$ is linearly independent, but at least one of $c_1, \dots, c_\ell, d$ is nonzero so this can't happen. Thus $d \neq 0$ and so

$$-dw = c_1 v_1 + \dots + c_\ell v_\ell$$

$$w = \left(-\frac{c_1}{d}\right) v_1 + \dots + \left(-\frac{c_\ell}{d}\right) v_\ell \quad \text{since } d \neq 0$$

so indeed $w$ is in $\text{span}\{v_1, \dots, v_\ell\}$. $\boxed{}$

## Theorem

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. In other words if $\{v_1, \ldots, v_l\}$ is a set of vectors in $\mathbb{R}^n$, then $\{v_1, \ldots, v_l\}$ is linearly dependent if $l > n$.

## Proof

$\{v_1, \ldots, v_l\}$ is linearly independent if $Ax = 0$ has only the trivial solution where $A = [v_1 | \cdots | v_l]$. However this impossible since $l > n$ as there are more variables than columns and so there must be a free variable (since at most $n < l$ pivot columns).

## Warning!

This theorem says nothing about $\{v_1, \ldots, v_l\}$ in $\mathbb{R}^n$ if $l \leq n$.

For example, in $\mathbb{R}^3$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$$

is linearly independent

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} \right\}$$

is linearly dependent